Reply to "Comment on 'Mean first passage time for anomalous diffusion.'"

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Using the Laplace transform method we show an exact solution for the mean free passage time of a subdiffusive particle, thereby correcting the mistake in our previous paper [Phys. Rev E **62**, 6065 (2000)]. The time diverges at large t.

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Anomalous diffusion is a widely investigated phenomenon with an increasing number of applications in different areas of science (see Refs. [1], [2], and hundreds of references therein). Although the asymptotic solutions of the boundary value problems can be easily obtained by different methods, the derivation of an exact solution is a challenging task. From a few existing methods for the solution mention should be made of the use of Fox functions [3,4], integral transformations based on the continuum time random walk [5], and Laplace transformations [6]. The latter method which was successfully applied to the normal diffusion [7], offers such advantages as simplicity and a derivation of exact solutions by the inverse Laplace transformation. In our previous work [6] the exact solution of the Laplace transformed Fokker-Planck equation was obtained correctly; however, for the special case of subdiffusion the inverse Laplace transform was performed incorrectly which resulted in the wrong answer. This fact was pointed out by Yuste and Lindenberg [8] who, however, did not provide an exact expression for the mean free passage time. The latter is obtained in this comment, thereby correcting our previously published [6] erroneous result for this special case.

The mean first passage time T (MFPT) is the time needed for a stochastically moving particle to reach one of the two absorbing boundaries $x=0$ or $x=L$, when initially it was located at some point x_0 in the interval $(0,L)$. It can be shown [9] that T is expressed, in terms of the probability density function $P(x,t)$, as

$$T=\int_0^\infty dt S(t)=\int_0^\infty dt\int_0^L dx P(x,t), \tag{1}$$

where $S(t)$ is the so-called survival probability.

The function $P(x,t)$ can be found from the solution of the Fokker-Planck equation, which for the case of subdiffusion has the fractional form [1,2]

$$\frac{\partial^\alpha P}{\partial t^\alpha}=-F_\alpha\frac{\partial P}{\partial x}+D_\alpha\frac{\partial^2 P}{\partial x^2}+\frac{t^\alpha}{\Gamma(1+\alpha)}\delta(x), \tag{2}$$

with $\alpha<1$. The fractional derivative in Eq. (2) is understood in terms of the Riemann-Liouville integral [1,2]

$$\frac{\partial^\alpha P(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t\frac{P(x,\tau)}{(t-\tau)^\alpha}d\tau. \tag{3}$$

The time-Laplace transform of Eq. (2) converts this partial differential equation into the ordinary one,

$$D_\alpha\frac{\partial^2 P(x,s)}{\partial x^2}-F_\alpha\frac{\partial P(x,s)}{\partial x}=s^\alpha P(x,s)-s^{\alpha-1}\delta(x-x_0). \tag{4}$$

Equation (4) has been solved separately in two intervals, P_1 for $0\leqslant x\leqslant x_0$ and P_2 for $x_0\leqslant x\leqslant L$, with matching conditions at $x=x_0$ yielding to [6]

$$P_1(x,s)=\frac{\exp[F_\alpha(x-x_0)/2D_\alpha]\sinh[\gamma(L-x_0)]\sinh(\gamma x)}{s^{1-\alpha/2}\sinh(\gamma L)},$$
$$P_2(x,s)=\frac{\exp[F_\alpha(x-x_0)/2D_\alpha]\sinh[\gamma(L-x)]\sinh(\gamma x_0)}{s^{1-\alpha/2}\sinh(\gamma L)}, \tag{5}$$

where $\gamma=s^{\alpha/2}D_\alpha^{-1/2}$. For simplicity we consider hereafter only the zero drift case, $F_\alpha=0$. The calculations for the nonzero drift case, $F_\alpha\neq 0$, are quite similar but slightly more cumbersome. The Laplace transform of the survival probability $S(s)$ is defined as

$$S(s)=S_1(s)+S_2(s)=\int_0^{x_0}P_1(x,s)dx+\int_{x_0}^L P_2(x,s)dx. \tag{6}$$

On replacing Eq. (5) with $F_\alpha=0$ into Eq. (6) and performing integration, one obtains

$$S(s)=\frac{1}{s}\left[1-\frac{\cosh\left[\frac{\gamma}{2}(L-2x_0)\right]}{\cosh\left(\frac{\gamma L}{2}\right)}\right]. \tag{7}$$

Finally, substituting Eq. (7) into Eq. (1) yields

$$T=\frac{1}{2\pi i}\int_0^\infty dt\int_C\frac{ds\,\exp(st)}{s}\left[1-\frac{\cosh\left[\frac{\gamma}{2}(L-2x_0)\right]}{\cosh\left(\frac{\gamma L}{2}\right)}\right],$$

$$\gamma=\frac{s^{\alpha/2}}{\sqrt{D_\alpha}}, \tag{8}$$

where the contour C is the usual Bromwich contour for the inverse Laplace transform.

The full calculation of T presents no problem. One can perform the inverse Laplace transform numerically or analytically, bypassing the branch point $s=0$. To this end, one has [10] to transform the Bromwich contour, which is parallel to the imaginary axis, to the contour composed of two arcs of radius R centered at the origin, and of arch r, also with a center at origin, and, finally, to go to the limits $R\to\infty$ and $r\to 0$. In such a manner the Bromwich integral (8) rearranges to the usual integral which can be calculated using the steepest-descent approximation. We will show these calculations elsewhere, noting here only two important points.

(1) In order to find the asymptotic value of the mean free passage time, one has to find the asymptotic value of the integrand in Eq. (8) for $s\to 0$ (or $t\to\infty$), and then to perform an integration over t, which immediately gives

$$T_{t\to\infty}\approx t^{1-\alpha^{*}}, \tag{9}$$

i.e., since $\alpha<1$, the mean free passage time T for subdiffusion diverges at large t.

(2) For normal diffusion ($\alpha=1$) there is no branch point at $s=0$. The s integral for this case is included in the table of Ref. [10], and one finally obtains [6] the well-known result $T=x_0(L-x_0)/2D$.

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